Signal Precision and Optimal Bonus Schemes in Relational Contracting

Chang-Koo Chi   Trond E. Olsen

Department of Economics
Norwegian School of Economics

June 15, 2017

INTRODUCTION

- Principal-agent framework in a repeated relationship
- Hidden effort, risk neutral parties
- Observable but non-verifiable signal
- Standard approach: replace global IC for effort by local one
- Optimal bonus scheme takes a simple form; a reward for performance exceeding a hurdle
- What if the first-order approach (FOA) fails?
- We show here that if a generalized MLRP holds, then a bonus scheme with the same form but an *adjusted* hurdle is optimal.

Timeline of Repeated Moral Hazard

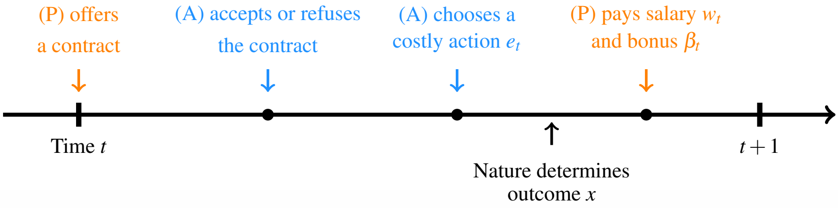


Figure: Sequence of Events in Period t

- Hidden effort $e \in [0, \bar{e}]$, private cost $c(e)$
- Signal $x \in \mathbb{X} \subset \Re^n$, density $f(x,e)$, and likelihood ratio $L(x,e) = \frac{f_e}{f}(x,e)$.
- Each party's expected payoff in period t is

$$u_t = w_t + \int_{\mathbb{X}} \beta_t(x) f(x,e_t)dx - c(e_t), \qquad \pi_t = B(e_t) - w_t - \int_{\mathbb{X}} \beta_t(x) f(x,e_t)dx.$$

- Total surplus in period t is $s(e_t) = B(e_t) - c(e_t)$.

Stationary Contract

Theorem 1 (Levin (2003)). *If an optimal contract exists, then there exists a stationary contract that is optimal.*

- We can focus on $\beta_t = \beta : \mathbb{X} \to \Re$ and $w_t = w$.
- Intuition: The two ways of providing incentives are equivalent under risk neutrality.

The optimal contract problem can be formulated as

$$\max_{\beta,e} \; s(e)$$

subject to

$$e \in \operatorname*{argmax}_{e'} \; w + \int_{\mathbb{X}} \beta(x) f(x,e') dx - c(e') \tag{IC}$$

and

$$0 \;\leq\; \beta(x) \;\leq\; \frac{\delta}{1-\delta} s(e). \tag{DE}$$

How do we solve this problem?

Utilizing the first-order approach, replace (IC) by FOC

$$c'(e) = \int \beta(x) f_e(x,e) dx.$$

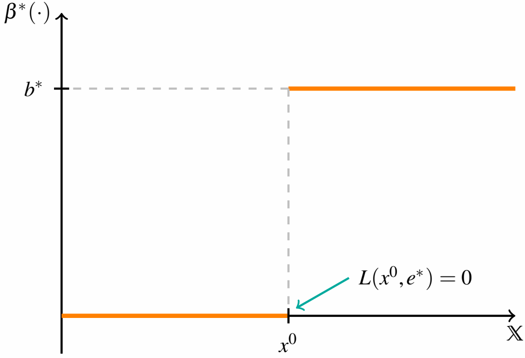
The Lagrangian is then linear with respect to β:

$$\mathfrak{L} = s(e) + \lambda \left[\int \beta(x) f_e(x,e) dx - c'(e) \right]$$

Consequently, the optimal bonus scheme is bang-bang with hurdle zero for the likelihood ratio:

$$\beta^*(x) = \begin{cases} 0 & \text{if } L(x,e^*) \leq 0 \\ b^* \equiv \frac{\delta}{1-\delta} s(e^*) & \text{otherwise.} \end{cases}$$

When the signal is unidimensional, the optimal payments are "one-step":



Example: FOA Fails

Let $x = e + \varepsilon, \quad \varepsilon \sim N(0, \sigma^2) \quad \Rightarrow \quad L(x, e^*) = \frac{x - e^*}{\sigma^2}$

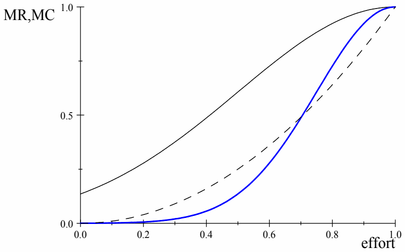
The expected payoff to the agent is

$$b \cdot \Pr\Big(L(X, e^*) > 0 \Big| e\Big) - c(e) \;=\; b \cdot \Pr\left(\frac{X - e^*}{\sigma^2} > 0 \middle| e\right) - c(e)$$
$$= b\left[1 - \Phi\left(\frac{e^* - e}{\sigma}\right)\right] - c(e)$$

Not globally concave with respect to e.

The marginal gain from effort is

$$b \cdot \Phi'\left(\frac{e^* - e}{\sigma}\right) \cdot \frac{1}{\sigma}$$



The previous example shows that FOA is not valid even with a natural signal structure.

When σ is small, a hurdle scheme provides strong marginal incentives at the given hurdle but possibly leads to a low expected payoff:

$$b \cdot \Pr\left(L(X, e^*) > 0 \middle| e^*\right) - c(e^*), \quad \text{with } b = \frac{\delta}{1-\delta} s(e^*).$$

Solution: Under some conditions, a lower hurdle (below the desired effort e^*) impairs marginal incentives but enhances the total payoff for effort.

GMLRP

Definition 2. *An information system satisfies the generalized monotone likelihood ratio property (GMLRP) if*

(i) *for any* $\kappa \in \Re$ *and efforts* $e, e' \in [0, \bar{e}]$, *there exists a* $\kappa' \in \Re$ *such that*

$$\left\{ x \in \mathbb{X} \,\middle|\, L(x,e) > \kappa \right\} \;=\; \left\{ x \in \mathbb{X} \,\middle|\, L(x,e') > \kappa' \right\}.$$

(ii) *for all* e, κ,

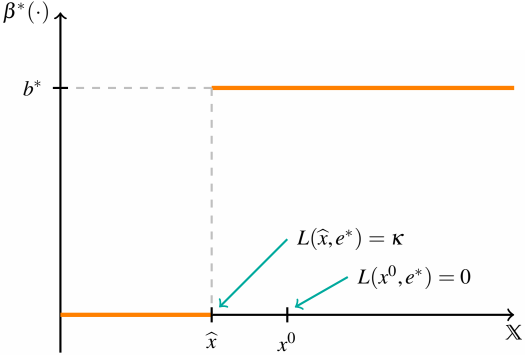
$$\Pr\left(L(X,e) > \kappa \,\middle|\, e' \right) \text{ is increasing in } e'.$$

Main Result 1

Proposition 1. *If GMLRP holds and no $e > e^{FB}$ is implementable, then the optimal bonus scheme takes a hurdle form:*

$$\beta^*(x) = \begin{cases} 0 & \text{if } L(x, e^*) \leq \kappa \\ b^* & \text{otherwise,} \end{cases}$$

$$\text{with } b^* = \frac{\delta}{1-\delta} s(e^*).$$



In particular, if $L_e < 0$, then $\kappa < 0$ and thus $\widehat{x} < x^0$ in the optimal contract.

Proof Idea

- For an arbitrary bonus scheme $0 \leq \beta(x) \leq b \; \forall \, x$ implementing e^*, there exists a hurdle scheme

$$\beta^* = \begin{cases} 0 & \text{if } L(x, e^*) \leq \kappa \\ b & \text{otherwise,} \end{cases}$$

such that the two schemes bring the agent the same expected payoff at $e = e^*$ but β^* induces a higher marginal incentive at $e = e^*$ than β.

- Then the hurdle scheme β^* would implement higher effort if

$$u\Big(\beta^*(X), e\Big) \;\leq\; u\Big(\beta(X), e\Big) \quad \forall \, e < e^*.$$

This inequality is indeed satisfied under GMLRP.

The optimal contract solves

$$\max_{e,\kappa} s(e)$$

subject to

$$e \in \underset{e'}{\operatorname{argmax}}\ b \cdot \Pr\Big(L(X,e) > \kappa \Big| e'\Big) - c(e'). \tag{IC'}$$

The optimal effort is the highest effort $e^* \in [0,\ \bar{e}]$ that satisfies

$$b \cdot \Pr\Big(L(X,e^*) > \kappa \Big| e\Big) - c(e) \ \leq\ b \cdot \Pr\Big(L(X,e^*) > \kappa \Big| e^*\Big) - c(e^*)$$

for all $e < e^*$.

Example: Revisited

Let $x = e + \varepsilon, \quad \varepsilon \sim N(0, \sigma^2)$.

$$L(x,e) = \frac{x-e}{\sigma^2}$$

The likelihood ratio $L(x,e)$ is increasing in x (MLRP) but decreasing in e.

Consider a hurdle scheme

$$\beta^* = \begin{cases} 0 & \text{if } L(x,e^*) \leq \kappa \\ b & \text{otherwise,} \end{cases}$$

$L(x,e^*) > \kappa$ iff $x - e^* > \hat{x}$, where

$$\hat{x} = \sigma^2 \kappa, \quad \text{and } \hat{x} < 0 \text{ iff } \kappa < 0.$$

Hence

$$\Pr\Big(L(X,e^*) > \kappa \Big| e'\Big) = \Pr\Big(x - e^* > \hat{x} \Big| e'\Big) = 1 - \Phi\left(\frac{e^* - e + \hat{x}}{\sigma}\right).$$

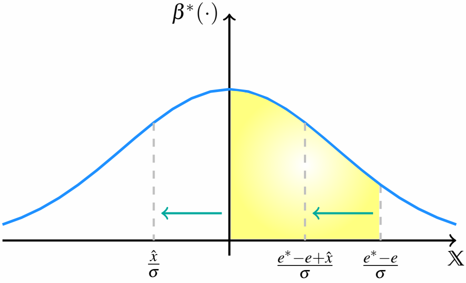
Example: Cont'd

Recall that the optimal effort is the highest e^* satisfying

$$b\left[1-\Phi\left(\frac{e^*-e+\hat{x}}{\sigma}\right)\right]-c(e) \;\leq\; b\left[1-\Phi\left(\frac{\hat{x}}{\sigma}\right)\right]-c(e^*) \quad \forall\, e<e^*$$

or

$$c(e^*)-c(e) \;\leq\; \Phi\left(\frac{e^*-e+\hat{x}}{\sigma}\right)-\Phi\left(\frac{\hat{x}}{\sigma}\right) \quad \forall\, e<e^*.$$

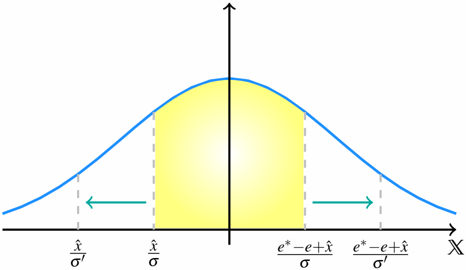


A lower hurdle $\hat{x}<0$ "relaxes" the global constraint and hence implements higher effort than $\hat{x}=0$.

Example 2: Cont'd

How does solution vary with σ?

$$c(e^*) - c(e) \;\leq\; \Phi\left(\frac{e^* - e + \hat{x}}{\sigma}\right) - \Phi\left(\frac{\hat{x}}{\sigma}\right) \quad \forall\, e < e^*.$$



A smaller variance $\sigma' < \sigma$ "relaxes" the global constraint and hence implements higher effort.

One can show that $e^* \to e^{FI}$ as $\sigma \to 0$.

Main Result 2

The next result generalizes the previous finding:

Proposition 2. *Suppose signal X is more precise than signal Y in Lehmann's sense: For each outcome y, there exists an increasing function $T_y : E \to \mathbb{X}$ such that*

$$F\Big(T_y(e), e\Big) = G(y, e). \tag{L}$$

Lehmann

Then for every hurdle scheme $\beta(y)$, there exists a hurdle scheme $\beta'(x)$ which implements higher effort.

Chi (2015): In the standard contracting framework (with a risk-averse agent) where FOA is valid, the condition (L) is a necessary and sufficient condition under which the principal implements an effort at a lower cost. If FOA fails, however, (L) is not even an sufficient condition.

Conclusion

- Under GMLRP, a hurdle scheme for the likelihood ratio is optimal even if FOA fails.
- The hurdle is less demanding when the likelihood ratio has a negative derivative with respect to effort.
- Shown an example where a smaller noise in the signal leads to higher effort (in the limit, converges to the full information effort).
- More generally, a higher degree of precision in the sense of Lehmann yields higher effort (and thus higher surplus).

Example: Gaussian Learning

Example 1. *Consider the two signals,* $X = e + \sigma_X \varepsilon$ *and* $Y = e + \sigma_Y \varepsilon$ *with*

$$\varepsilon \sim H(\cdot).$$

Then each distribution can be written

$$F(x,e) = H\left(\frac{x-e}{\sigma_X}\right) \qquad \text{and} \qquad G(y,e) = H\left(\frac{y-e}{\sigma_Y}\right).$$

Substitute $x = T_y(e)$ into F and set

$$H\left(\frac{T_y(e) - e}{\sigma_X}\right) = H\left(\frac{y-e}{\sigma_Y}\right).$$

Solving for T_y gives

$$T_y(e) = \frac{\sigma_X}{\sigma_Y}(y-e) + e,$$

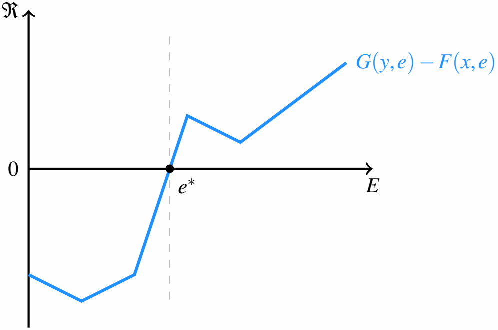
which is increasing in e whenever $\sigma_X < \sigma_Y$.

Go Back

CHARACTERIZATION

LEMMA 1 (CHI (2015)). *X is more precise than Y in Lehmann's sense if and only if for every $x \in \mathbb{X}$ and $y \in \mathbb{Y}$,*

$$G(y,e) - F(x,e) \text{ satisfies the single-crossing property in } e.$$



Go Back

Example 2. *Let*

$$[X] = \begin{bmatrix} 1/2 & 1/3 & 1/5 \\ 1/3 & 1/3 & 1/3 \\ 1/6 & 1/3 & 7/15 \end{bmatrix} \quad \textit{and} \quad [Y] = \begin{bmatrix} 5/12 & 1/3 & 1/4 \\ 1/3 & 1/3 & 1/3 \\ 1/4 & 1/3 & 5/12 \end{bmatrix},$$

where the ij-th element is $Pr(X = x_i | e = e_j)$.

$$[F(x,e)] = \begin{bmatrix} 1/2 & 1/3 & 1/5 \\ 5/6 & 2/3 & 8/15 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad [G(y,e)] = \begin{bmatrix} 5/12 & 1/3 & 1/4 \\ 3/4 & 2/3 & 7/12 \\ 1 & 1 & 1 \end{bmatrix}$$

It is easy to check that for every y and x, $G(y,e) - F(x,e)$ changes its sign at most once. Hence X is more precise than Y.

Go Back