

SIGNAL PRECISION AND OPTIMAL BONUS SCHEMES IN RELATIONAL CONTRACTING



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June 15, 2017

INTRODUCTION

- Principal-agent framework in a repeated relationship
- Hidden effort, risk neutral parties
- Observable but non-verifiable signal
- Standard approach: replace global IC for effort by local one
- Optimal bonus scheme takes a simple form; a reward for performance exceeding a hurdle
- What if the first-order approach (FOA) fails?
- We show here that if a generalized MLRP holds, then a bonus scheme with the same form but an *adjusted* hurdle is optimal.

TIMELINE OF REPEATED MORAL HAZARD

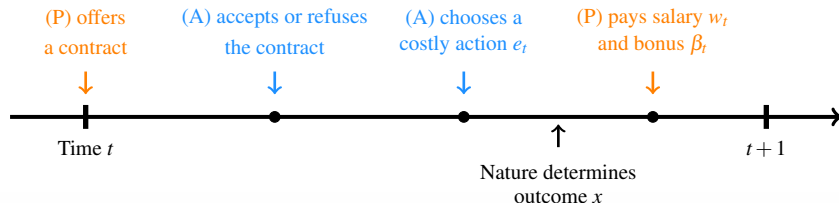


Figure: Sequence of Events in Period t

- Hidden effort $e \in [0, \bar{e}]$, private cost $c(e)$
- Signal $x \in \mathbb{X} \subset \mathfrak{R}^n$, density $f(x, e)$, and likelihood ratio $L(x, e) = \frac{f_e}{f}(x, e)$.
- Each party's expected payoff in period t is

$$u_t = w_t + \int_{\mathbb{X}} \beta_t(x) f(x, e_t) dx - c(e_t), \quad \pi_t = B(e_t) - w_t - \int_{\mathbb{X}} \beta_t(x) f(x, e_t) dx.$$

- Total surplus in period t is $s(e_t) = B(e_t) - c(e_t)$.

STATIONARY CONTRACT

THEOREM 1 (LEVIN (2003)). *If an optimal contract exists, then there exists a stationary contract that is optimal.*

- We can focus on $\beta_t = \beta : \mathbb{X} \rightarrow \mathbb{R}$ and $w_t = w$.
- **Intuition:** The two ways of providing incentives are equivalent under risk neutrality.

The optimal contract problem can be formulated as

$$\max_{\beta, e} s(e)$$

subject to

$$e \in \operatorname{argmax}_{e'} w + \int_{\mathbb{X}} \beta(x) f(x, e') dx - c(e') \quad (\text{IC})$$

and

$$0 \leq \beta(x) \leq \frac{\delta}{1 - \delta} s(e). \quad (\text{DE})$$

How do we solve this problem?

Utilizing the first-order approach, replace (IC) by FOC

$$c'(e) = \int \beta(x) f_e(x, e) dx.$$

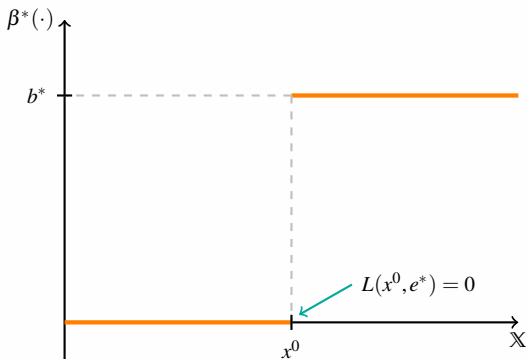
The Lagrangian is then linear with respect to β :

$$\mathcal{L} = s(e) + \lambda \left[\int \beta(x) f_e(x, e) dx - c'(e) \right]$$

Consequently, the optimal bonus scheme is bang-bang with hurdle zero for the likelihood ratio:

$$\beta^*(x) = \begin{cases} 0 & \text{if } L(x, e^*) \leq 0 \\ b^* \equiv \frac{\delta}{1-\delta} s(e^*) & \text{otherwise.} \end{cases}$$

When the signal is unidimensional, the optimal payments are "one-step":



EXAMPLE: FOA FAILS

Let $x = e + \varepsilon$, $\varepsilon \sim N(0, \sigma^2) \Rightarrow L(x, e^*) = \frac{x - e^*}{\sigma^2}$

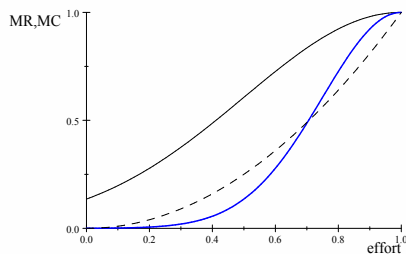
The expected payoff to the agent is

$$\begin{aligned} b \cdot \Pr(L(X, e^*) > 0 | e) - c(e) &= b \cdot \Pr\left(\frac{X - e^*}{\sigma^2} > 0 \mid e\right) - c(e) \\ &= b \left[1 - \Phi\left(\frac{e^* - e}{\sigma}\right) \right] - c(e) \end{aligned}$$

Not globally concave with respect to e .

The marginal gain from effort is

$$b \cdot \Phi'\left(\frac{e^* - e}{\sigma}\right) \cdot \frac{1}{\sigma}$$



The previous example shows that FOA is not valid even with a natural signal structure.

When σ is small, a hurdle scheme provides strong marginal incentives at the given hurdle but possibly leads to a low expected payoff:

$$b \cdot \Pr(L(X, e^*) > 0 | e^*) - c(e^*), \quad \text{with } b = \frac{\delta}{1 - \delta} s(e^*).$$

Solution: Under some conditions, a lower hurdle (below the desired effort e^*) impairs marginal incentives but enhances the total payoff for effort.

GMLRP

DEFINITION 2. *An information system satisfies the generalized monotone likelihood ratio property (GMLRP) if*

(i) *for any $\kappa \in \mathfrak{R}$ and efforts $e, e' \in [0, \bar{e}]$, there exists a $\kappa' \in \mathfrak{R}$ such that*

$$\{x \in \mathbb{X} \mid L(x, e) > \kappa\} = \{x \in \mathbb{X} \mid L(x, e') > \kappa'\}.$$

(ii) *for all e, κ ,*

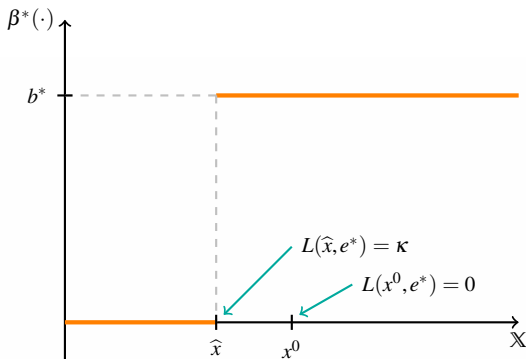
$\Pr(L(X, e) > \kappa \mid e')$ *is increasing in e' .*

MAIN RESULT 1

PROPOSITION 1. *If GMLRP holds and no $e > e^{FB}$ is implementable, then the optimal bonus scheme takes a hurdle form:*

$$\beta^*(x) = \begin{cases} 0 & \text{if } L(x, e^*) \leq \kappa \\ b^* & \text{otherwise,} \end{cases}$$

$$\text{with } b^* = \frac{\delta}{1 - \delta} s(e^*).$$



In particular, if $L_e < 0$, then $\kappa < 0$ and thus $\hat{x} < x^0$ in the optimal contract.

PROOF IDEA

- For an arbitrary bonus scheme $0 \leq \beta(x) \leq b \forall x$ implementing e^* , there exists a hurdle scheme

$$\beta^* = \begin{cases} 0 & \text{if } L(x, e^*) \leq \kappa \\ b & \text{otherwise,} \end{cases}$$

such that the two schemes bring the agent the same expected payoff at $e = e^*$ but β^* induces a higher marginal incentive at $e = e^*$ than β .

- Then the hurdle scheme β^* would implement higher effort if

$$u(\beta^*(X), e) \leq u(\beta(X), e) \quad \forall e < e^*.$$

This inequality is indeed satisfied under GMLRP.

The optimal contract solves

$$\max_{e, \kappa} s(e)$$

subject to

$$e \in \operatorname{argmax}_{e'} b \cdot \Pr(L(X, e) > \kappa | e') - c(e'). \quad (\text{IC}')$$

The optimal effort is the highest effort $e^* \in [0, \bar{e}]$ that satisfies

$$b \cdot \Pr(L(X, e^*) > \kappa | e) - c(e) \leq b \cdot \Pr(L(X, e^*) > \kappa | e^*) - c(e^*)$$

for all $e < e^*$.

EXAMPLE: REVISITED

Let $x = e + \varepsilon$, $\varepsilon \sim N(0, \sigma^2)$.

$$L(x, e) = \frac{x - e}{\sigma^2}$$

The likelihood ratio $L(x, e)$ is increasing in x (MLRP) but decreasing in e .

Consider a hurdle scheme

$$\beta^* = \begin{cases} 0 & \text{if } L(x, e^*) \leq \kappa \\ b & \text{otherwise,} \end{cases}$$

$L(x, e^*) > \kappa$ iff $x - e^* > \hat{x}$, where

$$\hat{x} = \sigma^2 \kappa, \quad \text{and } \hat{x} < 0 \text{ iff } \kappa < 0.$$

Hence

$$\Pr(L(X, e^*) > \kappa \mid e') = \Pr(x - e^* > \hat{x} \mid e') = 1 - \Phi\left(\frac{e^* - e + \hat{x}}{\sigma}\right).$$

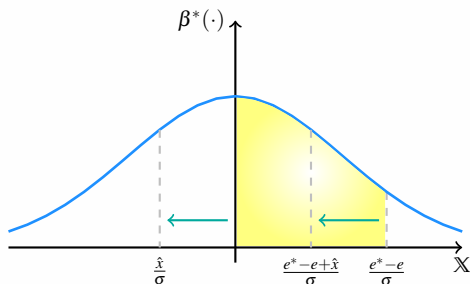
EXAMPLE: CONT'D

Recall that the optimal effort is the highest e^* satisfying

$$b \left[1 - \Phi \left(\frac{e^* - e + \hat{x}}{\sigma} \right) \right] - c(e) \leq b \left[1 - \Phi \left(\frac{\hat{x}}{\sigma} \right) \right] - c(e^*) \quad \forall e < e^*$$

or

$$c(e^*) - c(e) \leq \Phi \left(\frac{e^* - e + \hat{x}}{\sigma} \right) - \Phi \left(\frac{\hat{x}}{\sigma} \right) \quad \forall e < e^*.$$

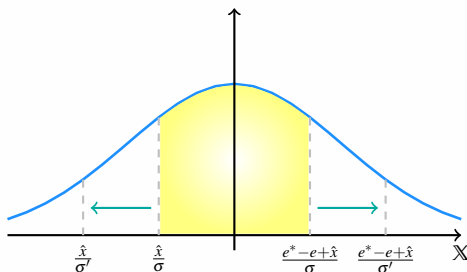


A lower hurdle $\hat{x} < 0$ "relaxes" the global constraint and hence implements higher effort than $\hat{x} = 0$.

EXAMPLE 2: CONT'D

How does solution vary with σ ?

$$c(e^*) - c(e) \leq \Phi\left(\frac{e^* - e + \hat{x}}{\sigma}\right) - \Phi\left(\frac{\hat{x}}{\sigma}\right) \quad \forall e < e^*.$$



A smaller variance $\sigma' < \sigma$ "relaxes" the global constraint and hence implements higher effort.

One can show that $e^* \rightarrow e^{FI}$ as $\sigma \rightarrow 0$.

MAIN RESULT 2

The next result generalizes the previous finding:

PROPOSITION 2. *Suppose signal X is more precise than signal Y in Lehmann's sense: For each outcome y , there exists an increasing function $T_y : E \rightarrow \mathbb{X}$ such that*

$$F\left(T_y(e), e\right) = G(y, e). \quad (\mathbf{L})$$

► Lehmann

Then for every hurdle scheme $\beta(y)$, there exists a hurdle scheme $\beta'(x)$ which implements higher effort.

Chi (2015): In the standard contracting framework (with a risk-averse agent) where FOA is valid, the condition **(L)** is a necessary and sufficient condition under which the principal implements an effort at a lower cost. If FOA fails, however, **(L)** is not even an sufficient condition.

CONCLUSION

- Under GMLRP, a hurdle scheme for the likelihood ratio is optimal even if FOA fails.
- The hurdle is less demanding when the likelihood ratio has a negative derivative with respect to effort.
- Shown an example where a smaller noise in the signal leads to higher effort (in the limit, converges to the full information effort).
- More generally, a higher degree of precision in the sense of Lehmann yields higher effort (and thus higher surplus).

EXAMPLE: GAUSSIAN LEARNING

EXAMPLE 1. Consider the two signals, $X = e + \sigma_X \varepsilon$ and $Y = e + \sigma_Y \varepsilon$ with

$$\varepsilon \sim H(\cdot).$$

Then each distribution can be written

$$F(x, e) = H\left(\frac{x - e}{\sigma_X}\right) \quad \text{and} \quad G(y, e) = H\left(\frac{y - e}{\sigma_Y}\right).$$

Substitute $x = T_y(e)$ into F and set

$$H\left(\frac{T_y(e) - e}{\sigma_X}\right) = H\left(\frac{y - e}{\sigma_Y}\right).$$

Solving for T_y gives

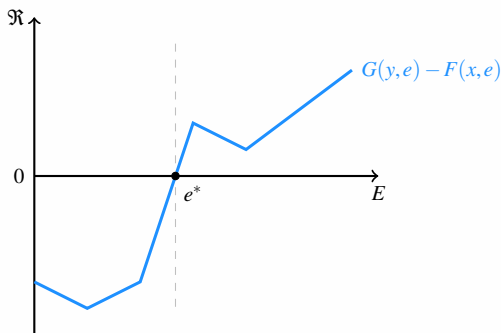
$$T_y(e) = \frac{\sigma_X}{\sigma_Y}(y - e) + e,$$

which is increasing in e whenever $\sigma_X < \sigma_Y$.

CHARACTERIZATION

LEMMA 1 (CHI (2015)). *X is more precise than Y in Lehmann's sense if and only if for every $x \in \mathbb{X}$ and $y \in \mathbb{Y}$,*

$G(y, e) - F(x, e)$ satisfies the single-crossing property in e .



EXAMPLE 2. *Let*

$$[X] = \begin{bmatrix} 1/2 & 1/3 & 1/5 \\ 1/3 & 1/3 & 1/3 \\ 1/6 & 1/3 & 7/15 \end{bmatrix} \quad \text{and} \quad [Y] = \begin{bmatrix} 5/12 & 1/3 & 1/4 \\ 1/3 & 1/3 & 1/3 \\ 1/4 & 1/3 & 5/12 \end{bmatrix},$$

where the ij -th element is $\Pr(X = x_i | e = e_j)$.

$$[F(x, e)] = \begin{bmatrix} 1/2 & 1/3 & 1/5 \\ 5/6 & 2/3 & 8/15 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad [G(y, e)] = \begin{bmatrix} 5/12 & 1/3 & 1/4 \\ 3/4 & 2/3 & 7/12 \\ 1 & 1 & 1 \end{bmatrix}$$

It is easy to check that for every y and x , $G(y, e) - F(x, e)$ changes its sign at most once. Hence X is more precise than Y .

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